

The completion of optimal $(3, 4)$ -packings*

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Abstract

A $3-(n, 4, 1)$ packing design consists of an n -element set X and a collection of 4-element subsets of X , called *blocks*, such that every 3-element subset of X is contained in at most one block. The packing number of quadruples $d(3, 4, n)$ denotes the number of blocks in a maximum $3-(n, 4, 1)$ packing design, which is also the maximum number $A(n, 4, 4)$ of codewords in a code of length n , constant weight 4, and minimum Hamming distance 4. In this paper the undecided 21 packing numbers $A(n, 4, 4)$ are shown to be equal to Johnson bound $J(n, 4, 4)$ ($= \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \lfloor \frac{n-2}{2} \rfloor \rfloor$) where $n = 6k + 5$, $k \in \{m : m \text{ is odd}, 3 \leq m \leq 35, m \neq 17, 21\} \cup \{45, 47, 75, 77, 79, 159\}$.

Keywords: constant weight code, packing design, candelabra system, s -fan design.

1 Introduction

A $3-(n, 4, 1)$ *packing design* consists of an n -element set X and a collection of 4-element subsets of X , called *blocks*, such that every 3-element subset of X is contained in at most one of them. Such a design is called a *packing quadruple* and denoted by $\text{PQS}(n)$ (as in [12]).

A $\text{PQS}(n)$ (X, \mathcal{A}) is called *maximum* if there does not exist any $\text{PQS}(n)$ (X, \mathcal{B}) with $|\mathcal{A}| < |\mathcal{B}|$, and shortly denoted by $\text{MPQS}(n)$. The packing number is the number of blocks in an $\text{MPQS}(n)$ and denoted by $d(3, 4, n)$, and by $A(n, 4, 4)$, where $A(n, d, w)$ is the maximum number of codewords in a code of length n , constant weight w , and minimum Hamming distance d .

The problem of determining $A(n, 4, 4)$ has received a lot of attention from the point of view of combinatorics and coding theory.

It is known that the Johnson bound $J(n, 4, 4)$ for the packing numbers [16] is given by

$$A(n, 4, 4) \leq J(n, 4, 4) = \begin{cases} \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \lfloor \frac{n-2}{2} \rfloor \rfloor & n \not\equiv 0 \pmod{6}, \\ \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \lfloor \frac{n-2}{2} \rfloor \rfloor - 1 \rfloor & n \equiv 0 \pmod{6}. \end{cases}$$

Here, $\lfloor x \rfloor$ denotes the largest integer not more than x .

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When $n \equiv 2, 4 \pmod{6}$, Hanani [7] showed that $A(n, 4, 4) = J(n, 4, 4)$ by constructing a PQS(n) with the property that each triple is contained in exactly one block. Such a design is called a *Steiner quadruple system* of order n and denoted by SQS(n). Deleting one point and all blocks containing it from an SQS($n+1$) yields that $A(n, 4, 4) = J(n, 4, 4)$ if $n \equiv 1, 3 \pmod{6}$. Brouwer [3] showed $A(n, 4, 4) = J(n, 4, 4)$ for $n \equiv 0 \pmod{6}$. The second author showed that $A(n, 4, 4) = J(n, 4, 4)$ for $n \equiv 5 \pmod{6}$ with 21 possible values [15]. These results are summarized as follows.

Theorem 1.1 [3, 7, 15] *For any positive integer $n \notin \{6k+5 : k = 3, 5, 7, 9, 11, 13, 15, 19, 23, 25, 27, 29, 31, 33, 35, 45, 47, 75, 77, 79, 159\}$, $A(n, 4, 4) = J(n, 4, 4)$.*

The purpose of this paper is to determine the last 21 undecided packing numbers $A(n, 4, 4)$. Throughout the remainder of this paper, an MPQS(n) is always assumed to have $J(n, 4, 4)$ blocks.

The rest of this paper is arranged as follows. In Section 2, we construct an MPQS(n) for $n \in \{23, 35, 47, 59, 71\}$ directly. In Section 3, we describe recursive constructions for MPQS(n)'s via candelabra quadruple systems. In Section 4 we determine the last 21 undecided packing numbers $A(n, 4, 4)$. Combining these results with Theorem 1.1, the packing numbers $A(n, 4, 4)$ are then completely determined.

2 Small values

In this section we construct an MPQS(n) for $n \in \{23, 35, 47, 59, 71\}$.

Lemma 2.1 *There is an MPQS(23).*

Proof: Let $X = \{0, 1, 2, \dots, 22\}$ and let α be a permutation as follows.

$$\alpha = (0\ 1)(2\ 3\ 4)(5\ 6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15\ 16)(17\ 18\ 19\ 20\ 21\ 22)$$

The following base blocks generate the required $J(23, 4, 4) = 419$ blocks under the action of the permutation α , where the first one base block generates only two distinct blocks and each of the other five base blocks in the first row generates three distinct blocks.

0 5 7 9	0 1 5 8	0 1 11 14	0 1 17 20	3 4 5 8	5 6 8 9
0 2 3 6	0 2 5 10	0 2 7 11	0 2 9 12	0 2 13 14	0 2 15 17
0 2 16 19	0 2 18 22	0 2 20 21	0 5 6 20	0 5 12 22	0 5 13 17
0 5 16 18	0 5 19 21	0 6 8 19	0 6 11 18	0 6 12 13	0 6 15 21
0 6 16 22	0 11 13 22	0 12 14 19	2 3 11 17	2 3 12 22	2 3 13 18
2 5 7 19	2 5 9 16	2 5 11 22	2 5 12 14	2 5 13 20	2 5 15 18
2 5 17 21	2 6 7 12	2 6 10 21	2 6 11 20	2 6 14 16	2 6 17 22
2 6 18 19	2 7 13 17	2 7 14 21	2 7 15 16	2 7 20 22	2 11 12 19
2 12 16 18	5 6 7 16	5 6 12 21	5 6 13 18	5 6 14 17	5 6 19 22
5 7 11 13	5 7 14 15	5 7 20 21	5 8 11 18	5 8 12 20	5 8 13 19
5 11 14 16	5 11 17 19	5 12 15 16	5 13 16 21	5 14 18 22	5 15 19 20
5 17 20 22	11 12 13 17	11 12 18 21	11 13 20 21	11 14 17 21	11 17 18 22
11 19 20 22					

□

The following lemma was proved by Stern and Lenz in [20].

Theorem 2.2 [20] *Let $G(L)$ be a graph with vertex set Z_{2k} where L is a set of integers in the range $1, 2, \dots, k$, such that $\{a, b\}$ is an edge of $G(L)$ if and only if $|b - a| \in L$, where $|b - a| = b - a$ if $0 \leq b - a \leq k$ and $|b - a| = a - b$ if $k < b - a < 2k$. Then $G(L)$ has a one-factorization if and only if $2k/\gcd(j, 2k)$ is even for some $j \in L$.*

Lemma 2.3 *There is an MPQS(35).*

Proof: We shall construct an MPQS(35) on $Z_{24} \cup \{x_1, x_2, \dots, x_{11}\}$. Beside the blocks of an MPQS(11) on $\{x_1, x_2, \dots, x_{11}\}$, the other blocks are divided into two parts described below.

For $1 \leq i \leq 11$ with $i \neq 8, 12$, let $\{F_i, F_{24-i}\}$ be a one-factorization of the graph $G(\{i\})$ over Z_{24} , and let F_{12} be the one-factor of the graph $G(\{12\})$ over Z_{24} . These one-factorizations exist by Theorem 2.2.

Let A be an 11×11 array as follows.

2	12	1	23	3	21	4	20	10	14	22
12	22	23	1	21	3	20	4	14	10	2
1	23	4	12	2	22	7	17	6	18	20
23	1	12	20	22	2	17	7	18	6	4
3	21	2	22	5	12	9	15	1	23	19
21	3	22	2	12	19	15	9	23	1	5
4	20	7	17	9	15	6	12	2	22	18
20	4	17	7	15	9	12	18	22	2	6
10	14	6	18	1	23	2	22	9	12	15
14	10	18	6	23	1	22	2	12	15	9
22	2	20	4	19	5	18	6	15	9	12

The first part consists of the following blocks:

$$\{x_i, x_j, a, b\}, \quad 1 \leq i < j \leq 11, \quad \{a, b\} \in F_{A(i,j)}.$$

The blocks in the second part are generated by the following base blocks modulo 24.

x_1	0	5	11	x_1	0	7	15	x_2	0	6	11	x_2	0	8	15	x_3	0	3	11	x_3	0	5	14
x_4	0	8	11	x_4	0	9	14	x_5	0	4	11	x_5	0	6	14	x_6	0	7	11	x_6	0	8	14
x_7	0	1	11	x_7	0	3	8	x_8	0	10	11	x_8	0	5	8	x_9	0	3	7	x_9	0	5	13
x_{10}	0	4	7	x_{10}	0	8	13	x_{11}	0	3	13	x_{11}	0	1	8	0	1	2	5	0	1	3	17
0	1	6	10	0	1	7	18	0	1	9	21	0	1	13	15	0	1	16	20	0	1	19	22
0	2	4	15	0	2	6	8	0	2	7	9	0	2	10	14	0	3	9	15	0	3	14	18
0	5	10	17																				

It is easy to check that the obtained blocks have no common triples. So, these blocks form a PQS(35). Further, it has $35 + \binom{11}{2} \times 12 + 37 \times 24 = 1583 = J(35, 4, 4)$ blocks and this PQS(35) is also optimal. Here, we also list the triples that are not contained in any block so that this construction of an MPQS(35) is more readable.

$\{x_i, a, b\},$	where $\{a, b\} \in F_{A(i,i)}$ and $1 \leq i \leq 11$
$\{k, k+8, k+16\},$	where $0 \leq k \leq 7$
$\{j, j+1, j+12\}, \{j, j+3, j+10\},$	where $j \in Z_{24}$
unused triples of an MPQS(11)	on $\{x_1, x_2, \dots, x_{11}\}.$

□

Lemma 2.4 *There is an MPQS(47).*

Proof: We shall construct an MPQS(47) on $Z_{36} \cup \{x_1, x_2, \dots, x_{11}\}$. Beside the blocks of an MPQS(11) on $\{x_1, x_2, \dots, x_{11}\}$, the other blocks are divided into two parts described below.

For $1 \leq i \leq 18$ with $i \neq 4, 8, 12, 16, 18$, let $\{F_i, F_{36-i}\}$ be a one-factorization of the graph $G(\{i\})$ over Z_{36} , and let F_{18} be the one-factor of the graph $G(\{18\})$ over Z_{36} . These one-factorizations exist by Theorem 2.2.

Let A be an 11×11 array as follows.

1	18	2	34	3	33	5	31	6	30	35
18	35	34	2	33	3	31	5	30	6	1
2	34	5	18	1	35	3	33	10	26	31
34	2	18	31	35	1	33	3	26	10	5
3	33	1	35	9	18	6	30	2	34	27
33	3	35	1	18	27	39	6	34	2	9
5	31	3	33	6	30	10	18	7	29	26
31	5	33	3	39	6	18	26	29	7	10
6	30	10	26	2	34	7	29	14	18	22
30	6	26	10	34	2	29	7	18	22	14
35	1	31	5	27	9	26	10	22	14	18

The first part consists of the following blocks:

$$\{x_i, x_j, a, b\}, \quad 1 \leq i < j \leq 11, \quad \{a, b\} \in F_{A(i,j)}.$$

The blocks in the second part are generated by the following base blocks modulo 36, where the underlined base block generates 18 distinct blocks.

x_1 0 4 14	x_1 0 7 19	x_1 0 8 21	x_1 0 9 20	x_2 0 10 14	x_2 0 12 19
x_2 0 13 21	x_2 0 11 20	x_3 0 4 13	x_3 0 6 17	x_3 0 7 21	x_3 0 8 20
x_4 0 9 13	x_4 0 11 17	x_4 0 14 21	x_4 0 12 20	x_5 0 4 17	x_5 0 5 16
x_5 0 7 15	x_5 0 10 22	x_6 0 13 17	x_6 0 11 16	x_6 0 8 15	x_6 0 12 22
x_7 0 1 9	x_7 0 2 16	x_7 0 4 19	x_7 0 11 23	x_8 0 8 9	x_8 0 14 16
x_8 0 15 19	x_8 0 12 23	x_9 0 1 17	x_9 0 3 12	x_9 0 4 15	x_9 0 5 13
x_{10} 0 16 17	x_{10} 0 9 12	x_{10} 0 11 15	x_{10} 0 8 13	x_{11} 0 3 15	x_{11} 0 2 32
x_{11} 0 7 23	x_{11} 0 8 25	<u>0 5 18 23</u>	0 1 2 19	0 2 4 20	0 1 3 29
0 1 4 5	0 1 6 10	0 1 7 25	0 1 8 34	0 1 11 13	0 1 12 15
0 1 14 31	0 1 16 22	0 1 21 27	0 1 23 30	0 1 24 26	0 2 5 7
0 2 8 30	0 2 9 29	0 2 11 21	0 2 14 23	0 2 15 24	0 2 17 27
0 3 6 31	0 3 7 11	0 3 8 32	0 3 9 14	0 3 13 16	0 3 17 20
0 3 21 30	0 4 9 28	0 4 10 16	0 4 11 22	0 4 18 24	0 5 10 17
0 5 14 22	0 5 15 20	0 6 13 23	0 8 16 26		

It is easy to check that the obtained blocks have no common triples. So, these blocks form a PQS(47). Further, it has $35 + \binom{11}{2} \times 18 + 81 \times 36 + 18 = 3959 = J(47, 4, 4)$ blocks and this PQS(47) is also optimal. Here, we also list the triples that are not contained in any block so that this construction of an MPQS(47) is more readable.

$$\begin{array}{ll} \{x_i, a, b\}, & \text{where } \{a, b\} \in F_{A(i,i)} \text{ and } 1 \leq i \leq 11, \\ \{k, k+12, k+24\}, & \text{where } 0 \leq k \leq 11, \\ \{j, j+3, j+18\}, \{j, j+2, j+6\}, & \{j, j+7, j+20\}, \{j, j+8, j+19\}, \text{ where } j \in Z_{36}, \\ \text{unused triples of an MPQS(11)} & \text{on } \{x_1, x_2, \dots, x_{11}\}. \end{array}$$

□

Let (X, \mathcal{B}) be a PQS(n). If there is an m -subset Y of X such that every triple of Y is not contained in any block, then such a PQS is called a *holey PQS* with a *hole* Y and denoted by HPQS(n, m).

Lemma 2.5 *There is an MPQS(59).*

Proof: We shall construct an MPQS(59) on $Z_{48} \cup \{x_1, x_2, \dots, x_{11}\}$. The required blocks are divided into four parts described below.

The first part consists of blocks of an MPQS(11) on $\{x_1, x_2, \dots, x_{11}\}$. For $j \in Z_4$, construct an HPQS(17, 5) on $\{4i + j : i \in Z_{12}\} \cup \{x_7, x_8, x_9, x_{10}, x_{11}\}$ with $\{x_7, x_8, x_9, x_{10}, x_{11}\}$ as a hole and with $J(17, 4, 4) - J(5, 4, 4) = 156$ blocks. Such a design exists by [15, Lemma 2.3]. The blocks of these four HPQS(17, 5) form the second part of blocks.

For $1 \leq i \leq 48$ with $i \neq 16, 24$, let $\{F_i, F_{48-i}\}$ be a one-factorization of the graph $G(\{i\})$ over Z_{48} , and let F_{24} be the one-factor of the graph $G(\{24\})$ over Z_{48} . These one-factorizations exist by Theorem 2.2.

Let A be an 11×11 array as follows, where some entries are empty.

3	24	4	44	6	42	1	47	2	46	45
24	3	44	4	42	6	47	1	46	2	3
4	44	5	24	8	40	2	46	1	47	43
44	4	24	43	40	8	46	2	47	1	5
6	42	8	40	10	24	3	45	5	43	38
42	6	40	8	24	38	45	3	43	5	10
1	47	2	46	3	45					
47	1	46	2	45	3					
2	46	1	47	5	43					
46	2	47	1	43	5					
45	3	43	5	38	10					

The third part consists of the following blocks:

$$\{x_i, x_j, a, b\}, \quad 1 \leq i < j \leq 11, (i, j) \notin \{(i', j') : 7 \leq i' < j' \leq 11\}, \quad \{a, b\} \in F_{A(i,j)}.$$

The blocks in the fourth part are generated by the following base blocks modulo 48.

x_1 0 5 12	x_1 0 8 22	x_1 0 9 27	x_1 0 10 25	x_1 0 11 28	x_1 0 13 29
x_2 0 7 12	x_2 0 14 22	x_2 0 18 27	x_2 0 15 25	x_2 0 17 28	x_2 0 16 29
x_3 0 3 9	x_3 0 7 26	x_3 0 10 28	x_3 0 11 23	x_3 0 13 27	x_3 0 15 31
x_4 0 6 9	x_4 0 19 26	x_4 0 18 28	x_4 0 12 23	x_4 0 14 27	x_4 0 16 31
x_5 0 1 12	x_5 0 2 16	x_5 0 4 25	x_5 0 7 22	x_5 0 9 28	x_5 0 13 30
x_6 0 11 12	x_6 0 14 16	x_6 0 21 25	x_6 0 15 22	x_6 0 19 28	x_6 0 17 30
x_7 0 5 11	x_7 0 7 25	x_7 0 9 22	x_7 0 10 27	x_7 0 14 29	x_8 0 6 11
x_8 0 18 25	x_8 0 13 22	x_8 0 17 27	x_8 0 15 29	x_9 0 3 21	x_9 0 6 19
x_9 0 7 17	x_9 0 9 23	x_9 0 11 26	x_{10} 0 18 21	x_{10} 0 13 19	x_{10} 0 10 17
x_{10} 0 14 23	x_{10} 0 15 26	x_{11} 0 1 15	x_{11} 0 2 23	x_{11} 0 6 13	x_{11} 0 9 26
x_{11} 0 11 29	0 1 5 6	0 1 7 8	0 1 9 10	0 1 11 13	0 1 14 17
0 1 16 18	0 1 19 20	0 1 21 22	0 1 23 26	0 1 31 33	0 1 32 35
0 1 36 38	0 2 5 7	0 2 6 8	0 2 9 11	0 2 10 15	0 2 14 20
0 2 19 21	0 2 22 28	0 2 30 36	0 2 35 40	0 3 7 40	0 3 8 37
0 3 10 39	0 3 11 44	0 3 12 41	0 3 13 38	0 3 14 43	0 3 15 18
0 3 19 22	0 3 20 23	0 4 9 43	0 4 10 14	0 4 11 39	0 4 13 41
0 4 17 21	0 4 18 22	0 4 19 23	0 5 16 26	0 5 17 22	0 5 18 23
0 5 20 33	0 5 21 28	0 5 25 32	0 5 27 37	0 6 14 37	0 6 15 21
0 6 16 22	0 6 17 40	0 6 23 29	0 7 16 39	0 7 18 34	0 7 21 37
0 8 17 29	0 8 18 26	0 8 21 33	0 8 23 35	0 8 27 39	0 10 22 36
0 1 2 25	0 2 4 26	0 3 6 27	0 5 10 29	0 6 12 30	0 7 14 31
0 9 18 33	0 10 20 34	0 11 22 35	0 1 3 4		

It is easy to check that the above blocks have no common triples. So, these blocks form a PQS(59). Further, it has $35 + 4 \times 156 + \left[\binom{11}{2} - \binom{5}{2}\right] \times 24 + 130 \times 48 = 7979 = J(59, 4, 4)$ blocks and this PQS(59) is also optimal. Here, we also list the triples that are not contained in any block so that this construction of an MPQS(59) is more readable.

$\{x_i, a, b\}$, where $\{a, b\} \in F_{A(i,i)}$ and $1 \leq i \leq 6$
 $\{j, j+14, j+15\}, \{j, j+21, j+23\}$, where $j \in Z_{48}$,
 $\{j, j+7, j+13\}, \{j, j+17, j+26\}$,
 $\{j, j+18, j+29\}$,
 unused triples of an MPQS(11) on $\{x_1, x_2, \dots, x_{11}\}$,
 unused triples of four HPQS(17, 5) on $\{4i+j : i \in Z_{12}\} \cup \{x_7, x_8, \dots, x_{11}\}$, $j \in Z_4$.

□

Lemma 2.6 *There is an MPQS(71).*

Proof: We shall construct an MPQS(71) on $Z_{48} \cup \{x_1, x_2, \dots, x_{23}\}$. The required blocks are divided into four parts described below.

The first part consists of blocks in an MPQS(23) on $\{x_1, x_2, \dots, x_{23}\}$. For $j \in Z_4$, construct an HPQS(17, 5) on $\{4i+j : i \in Z_{12}\} \cup \{x_{19}, x_{20}, x_{21}, x_{22}, x_{23}\}$ with $\{x_{19}, x_{20}, x_{21}, x_{22}, x_{23}\}$ as a hole and with $J(17, 4, 4) - J(5, 4, 4) = 156$ blocks. Such a design exists by [15, Lemma 2.3]. The blocks of these four HPQS(17, 5) form the second part of blocks.

For $1 \leq i \leq 48$ with $i \neq 16, 24$, let $\{F_i, F_{48-i}\}$ be a one-factorization of the graph $G(\{i\})$ over Z_{48} , and let F_{24} be the one-factor of the graph $G(\{24\})$ over Z_{24} . These one-factorizations exist by Theorem 2.2.

Let A be a 23×23 array as follows, where some entries are empty.

3	24	4	44	6	42	7	41	5	43	8	40	11	37	14	34	23	25	1	47	2	46	45
24	45	44	4	42	6	41	7	43	5	40	8	37	11	34	14	25	23	47	1	46	2	3
4	44	5	24	7	41	6	42	3	45	9	39	8	40	21	27	18	30	2	46	1	47	43
44	4	24	43	41	7	42	6	45	3	39	9	40	8	27	21	30	18	46	2	47	1	5
6	42	7	41	1	24	2	46	4	44	10	38	12	36	18	30	20	28	3	45	5	43	47
42	6	41	7	24	47	46	2	44	4	38	10	36	12	30	18	28	20	45	3	43	5	1
7	41	6	42	2	46	9	24	1	47	4	44	13	35	23	25	15	33	5	43	3	45	39
41	7	42	6	46	2	24	39	47	1	44	4	35	13	25	23	33	15	43	5	45	3	9
5	43	3	45	4	44	1	47	2	24	11	37	14	34	20	28	19	29	6	42	7	41	46
43	5	45	3	44	4	47	1	24	46	37	11	34	14	28	20	29	19	42	6	41	7	2
8	40	9	39	10	38	4	44	11	37	13	24	2	46	15	33	17	31	7	41	6	42	35
40	8	39	9	38	10	44	4	37	11	24	35	46	2	33	15	31	17	41	7	42	6	13
11	37	8	40	12	36	13	35	14	34	2	46	15	24	1	47	5	43	9	39	10	38	33
37	11	40	8	36	12	35	13	34	14	46	2	24	33	47	1	43	5	39	9	38	10	15
14	34	21	27	18	30	23	25	20	28	15	33	1	47	22	24	2	46	17	31	19	29	26
34	14	27	21	30	18	25	23	28	20	33	15	47	1	24	26	46	2	31	17	29	19	22
23	25	18	30	20	28	15	33	19	29	17	31	5	43	2	46	14	24	22	26	21	27	34
25	23	30	18	28	20	33	15	29	19	31	17	43	5	46	2	24	34	26	22	27	21	14
1	47	2	46	3	45	5	43	6	42	7	41	9	39	17	31	22	26					
47	1	46	2	45	3	43	5	42	6	41	7	39	9	31	17	26	22					
2	46	1	47	5	43	3	45	7	41	6	42	10	38	19	29	21	27					
46	2	47	1	43	5	45	3	41	7	42	6	38	10	29	19	27	21					
45	3	43	5	47	1	39	9	46	2	35	13	33	15	26	22	34	14					

The third part consists of the following blocks:

$$\{x_i, x_j, a, b\}, \quad 1 \leq i < j \leq 23, (i, j) \notin \{(i', j') : 19 \leq i' < j' \leq 23\}, \quad \{a, b\} \in F_{A(i, j)}.$$

The blocks in the fourth part are generated by the following base blocks modulo 48.

x_1 0 9 26	x_1 0 10 28	x_1 0 12 27	x_1 0 13 29	x_2 0 17 26	x_2 0 18 28
x_2 0 15 27	x_2 0 16 29	x_3 0 10 26	x_3 0 11 25	x_3 0 12 29	x_3 0 13 28
x_4 0 16 26	x_4 0 14 25	x_4 0 17 29	x_4 0 15 28	x_5 0 8 25	x_5 0 9 22
x_5 0 11 27	x_5 0 14 29	x_6 0 17 25	x_6 0 13 22	x_6 0 16 27	x_6 0 15 29
x_7 0 8 27	x_7 0 10 22	x_7 0 11 28	x_7 0 14 30	x_8 0 19 27	x_8 0 12 22
x_8 0 17 28	x_8 0 16 30	x_9 0 8 26	x_9 0 9 21	x_9 0 10 23	x_9 0 15 31
x_{10} 0 18 26	x_{10} 0 12 21	x_{10} 0 13 23	x_{10} 0 16 31	x_{11} 0 1 19	x_{11} 0 3 23
x_{11} 0 5 21	x_{11} 0 12 26	x_{12} 0 18 19	x_{12} 0 20 23	x_{12} 0 16 21	x_{12} 0 14 26
x_{13} 0 3 19	x_{13} 0 4 21	x_{13} 0 6 26	x_{13} 0 7 25	x_{14} 0 16 19	x_{14} 0 17 21
x_{14} 0 20 26	x_{14} 0 18 25	x_{15} 0 3 12	x_{15} 0 4 11	x_{15} 0 5 13	x_{15} 0 6 16
x_{16} 0 9 12	x_{16} 0 7 11	x_{16} 0 8 13	x_{16} 0 10 16	x_{17} 0 1 13	x_{17} 0 3 11
x_{17} 0 4 10	x_{17} 0 7 16	x_{18} 0 12 13	x_{18} 0 8 11	x_{18} 0 6 10	x_{18} 0 9 16
x_{19} 0 10 25	x_{19} 0 11 29	x_{19} 0 13 27	x_{20} 0 15 25	x_{20} 0 18 29	x_{20} 0 14 27
x_{21} 0 9 23	x_{21} 0 11 26	x_{21} 0 13 30	x_{22} 0 14 23	x_{22} 0 15 26	x_{22} 0 17 30
x_{23} 0 6 25	x_{23} 0 7 18	x_{23} 0 10 27	0 1 2 25	0 2 4 26	0 3 6 27
0 5 10 29	0 6 12 30	0 7 14 31	0 9 18 33	0 10 20 34	0 11 22 35
0 1 3 4	0 1 5 6	0 1 7 8	0 1 9 10	0 1 11 12	0 1 14 15
0 1 16 17	0 1 18 20	0 1 21 22	0 1 23 26	0 1 29 31	0 2 5 7
0 2 6 8	0 2 9 11	0 2 10 37	0 2 12 14	0 2 13 40	0 2 15 17
0 2 16 18	0 2 21 28	0 2 22 29	0 2 23 27	0 3 7 21	0 3 8 38
0 3 9 20	0 3 10 41	0 3 13 43	0 3 14 17	0 3 15 18	0 3 16 22
0 3 29 35	0 3 30 44	0 3 31 42	0 4 9 30	0 4 13 17	0 4 14 19
0 4 15 23	0 4 22 43	0 4 29 37	0 4 33 38	0 5 11 16	0 5 12 41
0 5 14 22	0 5 17 23	0 5 20 25	0 5 30 36	0 5 31 39	0 6 13 41
0 6 14 20	0 6 15 21	0 7 15 22	0 7 23 32	0 9 19 28	0 11 23 36

It is easy to check that the above blocks have no common triples. So, these blocks form a PQS(71). Further, it has $419 + 4 \times 156 + [(\binom{23}{2} - \binom{5}{2}) \times 24 + 150 \times 48 = 14075 = J(71, 4, 4)$ blocks and this PQS(71) is also optimal. Here, we also list the triples that are not contained in any block so that this construction of an MPQS(71) is more readable.

$\{x_i, a, b\}$,	where $\{a, b\} \in F_{A(i,i)}$ and $1 \leq i \leq 18$,
$\{j, j + 19, j + 25\}, \{j, j + 11, j + 18\}$,	$\{j, j + 17, j + 27\}$, where $j \in Z_{48}$,
unused triples of an MPQS(23)	on $\{x_1, x_2, \dots, x_{23}\}$,
unused triples of four HPQS(17, 5)	on $\{4i + j : i \in Z_{12}\} \cup \{x_{19}, x_{20}, \dots, x_{23}\}$, $j \in Z_4$.

□

3 Constructions for MPQSs

In this section we describe recursive constructions for MPQS(n)'s via candelabra quadruple systems.

Let v be a non-negative integer, let t be a positive integer and let K be a set of positive integers. A *candelabra t -system* (or *t -CS*) of order v , and block sizes from K is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

- (1) X is a set of v elements (called *points*).
- (2) S is a subset (called the *stem* of the candelabra) of X of size s .
- (3) $\mathcal{G} = \{G_1, G_2, \dots\}$ is a set of non-empty subsets (called *groups* or *branches*) of $X \setminus S$, which partition $X \setminus S$.
- (4) \mathcal{A} is a family of subsets (called *blocks*) of X , each of cardinality from K .
- (5) Every t -subset T of X with $|T \cap (S \cup G_i)| < t$ for all i , is contained in a unique block and no t -subsets of $S \cup G_i$ for all i , are contained in any block.

Such a system is denoted by $CS(t, K, v)$. By the *group type* (or *type*) of a t -CS $(X, S, \Gamma, \mathcal{A})$ we mean the list $(|G| | G \in \Gamma : |S|)$ of group sizes and stem size. The stem size is separated from the group sizes by a colon. If a t -CS has n_i groups of size g_i , $1 \leq i \leq r$, and stem size s , then we use the notation $(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$ to denote group type. A candelabra system with $t = 3$ and $K = \{4\}$ is called a *candelabra quadruple system* and briefly denoted by $CQS(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$. A $CS(t, K, v)$ with group type $(1^v : 0)$ is usually called a *t -wise balanced design* and shortly denoted by $S(t, K, v)$. As well, the group set \mathcal{G} and the stem S in the quadruple $(X, S, \mathcal{G}, \mathcal{A})$ can be omitted and we write (X, \mathcal{A}) instead of $(X, S, \mathcal{G}, \mathcal{A})$. When $K = \{k\}$, we simply write k instead of K .

Theorem 3.1 [18] *There is a $CQS(6^k : 0)$ for any $k \geq 0$.*

Theorem 3.2 [9, 11, 17] *A $CQS(g^3 : s)$ exists for all even s and all $g \equiv 0, s \pmod{6}$ with $g \geq s$.*

Theorem 3.3 [5, 21] *There exists a $CQS(g^4 : s)$ if and only if $g \equiv 0 \pmod{2}$, $s \equiv 0 \pmod{2}$ and $0 \leq s \leq 2g$.*

Theorem 3.4 [21] *A $CQS(g^5 : s)$ exists for all $g \equiv 0 \pmod{6}$, $s \equiv 0 \pmod{2}$ and $0 \leq s \leq 3g$.*

Lemma 3.5 [15] *There is a $CQS(12^k : 6)$ for any $k \geq 3$.*

With the aid of CQSs, a construction of MPQS(n) for $n \equiv 5 \pmod{6}$ has been stated in [15].

Construction 3.6 [15] *Suppose that there is a $CQS(g_0^1 g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r} : s)$, where $s \equiv 6 \pmod{12}$, $g_i \equiv 0 \pmod{12}$ for $1 \leq i \leq r$, and $g_0 \equiv 0 \pmod{6}$. If there is an MPQS($g_0 + s - 1$) and an HPQS($g_i + s - 1, s - 1$) with $J(g_i + s - 1, 4, 4) - J(s - 1, 4, 4)$ blocks for $1 \leq i \leq r$, then there is an MPQS($\sum_{1 \leq i \leq r} a_i g_i + g_0 + s - 1$).*

Similar to the proof of Construction 3.6, we can get another construction for $n \equiv 5 \pmod{6}$.

Construction 3.7 *Suppose that there is a $CQS(g_0^1 g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r} : s)$, where $s \equiv g_i \equiv 0 \pmod{12}$ for $0 \leq i \leq r$. If there is an MPQS($g_0 + s - 1$) and an HPQS($g_i + s - 1, s - 1$) with $J(g_i + s - 1, 4, 4) - J(s - 1, 4, 4)$ blocks for $1 \leq i \leq r$, then there is an MPQS($\sum_{1 \leq i \leq r} a_i g_i + g_0 + s - 1$).*

Proof: Let $(X, S, \mathcal{G}, \mathcal{B})$ be a given CQS($g_0^1 g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r} : s$). We shall construct the desired design as follows.

Take a point x from S and let $S' = S \setminus \{x\}$. Denote $\mathcal{B}' = \{B \in \mathcal{B} : x \notin B\}$. For a special group G with $|G| = g_0$, construct an MPQS($g_0 + s - 1$) on $G \cup S'$. Such a design exists by assumption. Denote its block set by \mathcal{C}_G . For each group $G' \neq G$, construct an HPQS($|G'| + s - 1, s - 1$) on $G' \cup S'$ with a hole S' and $J(|G'| + s - 1, 4, 4) - J(s - 1, 4, 4)$ blocks. Such a design exists by assumption. Denote its block set by $\mathcal{C}_{G'}$.

Let

$$\mathcal{A} = \mathcal{B}' \cup \mathcal{C}_G \cup \left(\bigcup_{G' \in \mathcal{G}, G' \neq G} \mathcal{C}_{G'} \right).$$

It is easy to see that all blocks in \mathcal{A} have no common triples. So, $(X \setminus \{x\}, \mathcal{A})$ is a PQS($\sum_{1 \leq i \leq r} a_i g_i + g_0 + s - 1$). It is left to check that $|\mathcal{A}| = J(\sum_{1 \leq i \leq r} a_i g_i + g_0 + s - 1, 4, 4)$.

Let $u = g_0 + \sum_{1 \leq i \leq r} a_i g_i$ and $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. Clearly, $\mathcal{B}' = \mathcal{B} \setminus \mathcal{B}_x$. Since \mathcal{B} is the block set of a CQS($g_0^1 g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r} : s$) and $\{B \setminus \{x\} : B \in \mathcal{B}_x\}$ is the block set of a GDD($2, 3, u$) of type $g_0^1 g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r}$, we have that $|\mathcal{B}| = \frac{1}{4}[(\binom{u+s}{3} - \binom{g_0+s}{3}) - \sum_{1 \leq i \leq r} a_i (\binom{g_i+s}{3} - \binom{s}{3})]$ and $|\mathcal{B}_x| = \frac{1}{3}[(\binom{u}{2} - \binom{g_0}{2}) - \sum_{1 \leq i \leq r} a_i \binom{g_i}{2}]$. By simple computing, we have

$$|\mathcal{B}'| = |\mathcal{B}| - |\mathcal{B}_x| = \frac{1}{24}[u^3 - g_0^3 - \sum_{1 \leq i \leq r} a_i g_i^3 + (3s - 7)(u^2 - g_0^2 - \sum_{1 \leq i \leq r} a_i g_i^2)].$$

By the definition, $J(n, 4, 4) = \frac{1}{24}[n^3 - 4n^2 + n - 18]$ for $n \equiv 11 \pmod{12}$. Since $|\mathcal{C}_{G'}| = J(|G'| + s - 1, 4, 4) - J(s - 1, 4, 4)$, $|G'| \equiv 0 \pmod{12}$ and $s - 1 \equiv 11 \pmod{12}$, we have $|\mathcal{C}_{G'}| = \frac{1}{24}[|G'|^3 + |G'|^2(3s - 7) + |G'|(3s^2 - 14s + 12)]$. So,

$$\left| \bigcup_{G' \in \mathcal{G}, G' \neq G} \mathcal{C}_{G'} \right| = \frac{1}{24} \sum_{1 \leq i \leq r} a_i [g_i^3 + g_i^2(3s - 7) + g_i(3s^2 - 14s + 12)].$$

Also,

$$|\mathcal{C}_G| = \frac{1}{24}[g_0^3 + g_0^2(3s - 7) + g_0(3s^2 - 14s + 12) + s^3 - 7s^2 + 12s - 24].$$

Since $|\mathcal{A}| = |\mathcal{B}'| + |\mathcal{C}_G| + |\bigcup_{G' \in \mathcal{G}, G' \neq G} \mathcal{C}_{G'}|$, the number of blocks is

$$\frac{1}{24}[u^3 + u^2(3s - 7) + u(3s^2 - 14s + 12) + s^3 - 7s^2 + 12s - 24],$$

which is equal to $J(u + s - 1, 4, 4)$. This completes the proof. \square

From Constructions 3.6-3.7 CQSs are useful in the constructions for MPQSs. A recursive construction for CQSs has been stated in [15].

Let v be a non-negative integer, let t be a positive integer and K be a set of positive integers. A *group divisible t -design* (or t -GDD) of order v and block sizes from K denoted by GDD(t, K, v) is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

- (1) X is a set of v elements (called *points*);
- (2) $\mathcal{G} = \{G_1, G_2, \dots\}$ is a set of non-empty subsets (called *groups*) of X , which partition X ;

(3) \mathcal{B} is a family of subsets (called *blocks*) of X each of cardinality from K such that each block intersects any given group in at most one point;

(4) each t -set of points from t distinct groups is contained in exactly one block.

The *type* of t -GDD is defined as the list $\{|G| : G \in \mathcal{G}\}$. When $K = \{k\}$, we simply write k for K .

A GDD(3, 4, v) of type r^m is called an H design (as in [19]) and denoted by $H(m, r, 4, 3)$.

Theorem 3.8 [14, 19] *For $m > 3$ and $m \neq 5$, an $H(m, r, 4, 3)$ exists if and only if rm is even and $r(m-1)(m-2)$ is divisible by 3. For $m = 5$, $H(5, r, 4, 3)$ exists if r is even, $r \neq 2$ and $r \not\equiv 10, 26 \pmod{48}$.*

Let $(X, S, \mathcal{G}, \mathcal{A})$ be a $CS(3, K, v)$ of type $(g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r} : s)$ with $s > 0$ and let $S = \{\infty_1, \dots, \infty_s\}$. For $1 \leq i \leq s$, let $\mathcal{A}_i = \{A \setminus \{\infty_i\} : A \in \mathcal{A}, \infty_i \in A\}$ and $\mathcal{A}_T = \{A \in \mathcal{A} : A \cap S = \emptyset\}$. Then the $(s+3)$ -tuple $(X, \mathcal{G}, \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_s, \mathcal{A}_T)$ is called an s -fan design (as in [10]). If block sizes of \mathcal{A}_i and \mathcal{A}_T are from $K_i (1 \leq i \leq s)$ and K_T , respectively, then the s -fan design is denoted by s -FG(3, $(K_1, K_2, \dots, K_s, K_T), \sum_{i=1}^r a_i g_i$) of type $g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r}$.

Below is a recursive construction for CQSs, which was obtained by applying Hartman's fundamental construction for 3-CSs [10],

Lemma 3.9 [15] *Suppose there is an e -FG(3, $(K_1, \dots, K_e, K_T), v$) of type $g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r}$ with $e \geq 1$, $K_i \subset \{k \geq 3 : k \text{ is an integer}\} (2 \leq i \leq e)$ and $K_T \subset \{k \geq 4 : k \text{ is an integer}\}$. Suppose that $b \equiv 0 \pmod{6}$ and there exists a CQS($b^{k_1} : s$) for any $k_1 \in K_1$. Then there exists a CQS($(bg_1)^{a_1} (bg_2)^{a_2} \cdots (bg_r)^{a_r} : b(e-1) + s$).*

In the next section, we shall obtain some CQSs and then determine the packing numbers $A(n, 4, 4)$.

4 Existence of MPQSs

In this section we shall determine the existence of the last 21 undecide MPQS(n) for $n \in \{6k+5 : k = 3, 5, 7, 9, 11, 13, 15, 19, 23, 25, 27, 29, 31, 33, 35, 45, 47, 75, 77, 79, 159\}$.

Lemma 4.1 *There is a CQS($24^k : 12$) for any $k \geq 3$.*

Proof: For $k \equiv 0, 1 \pmod{3}$, there is a 2-FG(3, $(3, 3, 4), 2k$) of type 2^k , which can be obtained by deleting two points from an SQS($2k+2$) in [7]. Applying Lemma 3.9 with $b = 12$ and the known CQS($12^3 : 0$) in Lemma 3.2 gives a CQS($24^k : 12$).

For $k \equiv 2 \pmod{3}$, there is a 2-FG(3, $(\{3, 5\}, \{3, 5\}, \{4, 6\}), 2k$) of type 2^k , which can be obtained by deleting two points from two distinct groups of a CQS($6^{(k+1)/3} : 0$) in Theorem 3.1. A CQS($24^k : 12$) is then obtained by applying Lemma 3.9 with $b = 12$ and the known CQS($12^j : 0$) ($j = 3, 5$) by Theorem 3.2 and Theorem 3.4. \square

Lemma 4.2 *There is an MPQS($24k+11$) for any $k \geq 3$. So, there is an MPQS(n) for $n \in \{6k+5 : k = 5, 13, 25, 29, 33, 45, 77\}$*

Proof: By Lemma 4.1, there is a CQS($24^k : 12$). Apply Construction 3.7 with $g_0 = g_1 = 24$, $r = 1$, $a_1 = k - 1$ and $s = 12$. Since there is an MPQS(35) and an HPQS(35, 11) with $J(35, 4, 4) - J(11, 4, 4)$ blocks which exists from the proof of Lemma 2.3, there is an MPQS($24k + 11$). \square

Lemma 4.3 *There is an MPQS($6k + 5$) for $k \in \{27, 35\}$.*

Proof: For $k = 27$, there is a CQS($48^3 : 24$) by Theorem 3.2. Since there is an MPQS(71) and an HPQS(71, 23) with $J(71, 4, 4) - J(23, 4, 4)$ blocks which exists from the proof of Lemma 2.6, there is an MPQS($6k + 5$) by Construction 3.7.

For $k = 35$, there is a CQS($48^4 : 24$) by Theorem 3.3. Since there is an MPQS(71) and an HPQS(71, 23) with $J(71, 4, 4) - J(23, 4, 4)$ blocks, there is an MPQS($6k + 5$) by Construction 3.7. \square

Lemma 4.4 *There is an MPQS(191).*

Proof: Deleting one point from an SQS(16) containing a subdesign S(2, 4, 16) [13, Theorem 1.3] gives a 1-FG(3, (3, 4), 15) of type 3^5 . Applying Lemma 3.9 with $b = 12$ and the known CQS($12^3 : 12$) gives a CQS($36^5 : 12$). Since there is an MPQS(47) and an HPQS(47, 11) with $J(47, 4, 4) - J(11, 4, 4)$ blocks which exists from the proof of Lemma 2.4, there is an MPQS(191) by Construction 3.7. \square

The next lemma is the well-known result on S(3, k, v)s.

Lemma 4.5 [6] *For any prime power q there exists an S(3, $q + 1, q^2 + 1$) and an S(3, 6, 22).*

Lemma 4.6 *There is an MPQS($6k + 5$) for $k \in \{19, 23\}$.*

Proof: Deleting two points of an S(3, 6, $k + 3$) by Lemma 4.5 gives a 2-FG(3, (5, 5, 6), $k + 1$) of type $4^{(k+1)/4}$. Further, deleting one point from a group give a 2-FG(3, ({4, 5}, {4, 5}, {4, 5, 6}), k) of type $4^{(k-3)/4}3^1$. Applying Lemma 3.9 with $b = 6$ and the known CQS($6^j : 0$) for $j \in \{4, 5\}$ in Theorem 3.1 gives a CQS($24^{(k-3)/4}18^1 : 6$). Since there is an HPQS(29, 5) with $J(29, 4, 4) - J(5, 4, 4)$ blocks [15, Lemma 4.4] and an MPQS(23) by Lemma 2.1, there is an MPQS($6k + 5$) by Construction 3.6. \square

Lemma 4.7 *There is an MPQS(95).*

Proof: Deleting one point of an S(3, 5, 17) by Lemma 4.5 gives a 1-FG(3, (4, 5), 16) of type 4^4 . Further, deleting one point from a group give a 1-FG(3, ({3, 4}, {4, 5}), 15) of type 4^33^1 . Applying Lemma 3.9 with $b = 6$ and the known CQS($6^j : 6$) for $j \in \{3, 4\}$ in Theorem 3.2 and Theorem 3.3 gives a CQS($24^318^1 : 6$). Since there is an HPQS(29, 5) with $J(29, 4, 4) - J(5, 4, 4)$ blocks [15, Lemma 4.4] and an MPQS(23) by Lemma 2.1, there is an MPQS(95) by Construction 3.6. \square

Lemma 4.8 *There is an MPQS($6k + 5$) for $k \in \{47, 75, 79, 159\}$.*

Proof: For $k = 47$, deleting two points from an $S(3, 8, 50)$ by Lemma 4.5 gives a 2-FG($3, (7, 7, 8), 48$) of type 6^8 . Further, deleting one point gives a 2-FG($3, (\{6, 7\}, \{6, 7\}, \{7, 8\}), 48$) of type $6^7 5^1$. Applying Lemma 3.9 with $b = 6$ and the known CQS($6^j : 0$) for $j \in \{6, 7\}$ by Theorem 3.1 gives a CQS($36^7 30^1 : 6$). Since there is an HPQS($41, 5$) with $J(41, 4, 4) - J(5, 4, 4)$ blocks by [15, Lemma 4.4] and an MPQS(35) by Lemma 2.3, there is an MPQS($6k + 5$) by Construction 3.6.

For $k = 75, 79$, deleting two points from an $S(3, 10, 82)$ by Lemma 4.5 gives a 2-FG($3, (9, 9, 10), 80$) of type 8^{10} . Further, deleting $80 - k$ points from one group gives a 2-FG($3, (\{8, 9\}, \{8, 9\}, \{8, 9, 10\}), 75$) of type $8^9(k - 72)^1$. Applying Lemma 3.9 with $b = 6$ and the known CQS($6^j : 0$) for $j \in \{8, 9\}$ by Theorem 3.1 gives a CQS($48^9(6k - 432)^1 : 6$). Since there is an HPQS($53, 5$) with $J(53, 4, 4) - J(5, 4, 4)$ blocks by [15, Lemma 4.4] and an MPQS(23) by Lemma 2.1 and an MPQS(47) by Lemma 2.4, there is an MPQS($6k + 5$) by Construction 3.6.

For $k = 159$, deleting two points from an $S(3, 14, 169)$ by Lemma 4.5 gives a 2-FG($3, (13, 13, 14), 168$) of type 12^{14} . Further, deleting nine points from one group gives a 2-FG($3, (\{12, 13\}, \{12, 13\}, \{12, 13, 14\}), 159$) of type $12^{13} 3^1$. Applying Lemma 3.9 with $b = 6$ and the known CQS($6^j : 0$) for $j \in \{12, 13\}$ by Theorem 3.1 gives a CQS($72^{13} 18^1 : 6$). Since there is an HPQS($77, 5$) with $J(77, 4, 4) - J(5, 4, 4)$ blocks by [15, Lemma 4.4] and an MPQS(23) by Lemma 2.1, there is an MPQS($6k + 5$) by Construction 3.6. \square

Combining Theorem 1.1, Lemmas 2.1-2.6, Lemmas 4.2-4.4 and Lemmas 4.6-4.8, we obtain the main result of this paper.

Theorem 4.9 *For any positive integer n , it holds that $A(n, 4, 4) = J(n, 4, 4)$.*

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